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Three additional families of Lagrangian surfaces of constant curvature in complex projective plane

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Abstract

Three additional families of Lagrangian surfaces of constant curvature in complex projective plane $CP^2(4)$ shall be added to the list of Theorem 1 in my article [B.-Y. Chen, Classification of Lagrangian surfaces of constant curvature in complex projective planes, J. Geom. Phys. 53 (2005) 428–460].

MSC: 53D12; 53C40; 53C42

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1. Three additional families

Beside (1)–(29), the following three families of Lagrangian surfaces of constant curvature shall be added to the list in Theorem 1 of my earlier article [1].

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(30) *Lagrangian surfaces of positive curvature* b^2 *defined by* $\pi \circ L$ *with*

$$L = ((\sin bs)^{1-\mathrm{i}ab^{-1}}, (\cos bs)^{1-\mathrm{i}ab^{-1}} z(t))$$

with $a = \sqrt{1-b^2}$, $b \in (0,1)$, *and* $z(t)$ *is a unit speed Legendre curve in* $S^3(1) \subset \mathbf{C}^2$.

(31) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$ *with*

$$L = \left(a \cosh bs + \mathrm{i}\sqrt{c^2 - a^2 \sinh^2 bs}\right)^{a/b} \times \left(\frac{z(t) \sinh bs}{\mathrm{e}^{\mathrm{i}b^{-1}c \tanh^{-1}(c \cosh bs/\sqrt{c^2-a^2 \sinh^2 bs})}}, \frac{\sqrt{b^2+c^2-\sinh^2 bs}\, \mathrm{e}^{-(\mathrm{i}a^2(b^2+c^2)/(b^2(a^2+c^2))) \cot^{-1}(b \cosh bs/\sqrt{c^2-a^2 \sinh^2 bs})}}{\sqrt{b^2+c^2}(a^2+c^2)^{a/2b}\, \mathrm{e}^{(\mathrm{i}c^2/(b^2(a^2+c^2))) \tan^{-1}(b \cosh bs/\sqrt{c^2-a^2 \sinh^2 bs})}} \right),$$

where c *is a positive number,* $a = \sqrt{1+b^2}$ *and* $z(t)$ *is a Legendre curve of constant speed* $(a^2+b^2)^{-a/2b}$ *in* $S^3(r) \subset \mathbf{C}^2$ *with radius* $r = (a^2+c^2)^{-a/2b}/\sqrt{b^2+c^2}$.

(32) *Lagrangian surfaces of negative curvature* $-b^2$ *defined by* $\pi \circ L$ *with*

$$L = \left(z(t) \left(a + \mathrm{i}\sqrt{c^2\, \mathrm{e}^{-2bs} - a^2}\right)^{a/b} \mathrm{e}^{(a+b)s - \mathrm{i}b^{-1}\sqrt{c^2\, \mathrm{e}^{-2bs} - a^2}}, \frac{(c^2 - \mathrm{e}^{2bs})\left(c^2 - 2\,\mathrm{e}^{2bs}\left(a^2 + \mathrm{i}a\sqrt{c^2\, \mathrm{e}^{-2bs} - a^2}\right)\right)^{a/2b}}{c^{(a+b)/b}\left(c^2 - \mathrm{e}^{2bs}\left(a^2 + b^2 + 2\mathrm{i}b\sqrt{c^2\, \mathrm{e}^{-2bs} - a^2}\right)\right)^{1/2}} \right),$$

where c *is a positive number,* $a = \sqrt{1+b^2}$ *and* $z(t)$ *is a Legendre curve with speed* $\mathrm{e}^{\theta(t)}\, c^{-a/b}$ *in* $S^3(r) \subset \mathbf{C}^2$, $r = c^{-1-a/b}$.

2. Remarks

We follow the same notation as [1]. For Case (I.ii.b.2.3) in article [1], the solution (5.56) of system (5.55) is degenerated whenever $c^2 = a^2$. So Case (23) in Theorem 1 of [1] occurs only under the condition $a^2 \neq c^2$.

When $a^2 = c^2$, system (5.55) reduces to

$$L_{ss} = \mathrm{i}a(\tan bs - \cot bs)Ls - L, \quad a = \sqrt{1-b^2}, \qquad L_{st} = (\mathrm{i}a - b)\tan bs L_t,$$
$$L_{tt} = (b + \mathrm{i}a)\sin bs \cos bs L_s + \mathrm{i} f(t) L_t - \cos^2 bs L. \tag{2.1}$$

After solving the first two equations of this PDE system we obtain

$$L = c_1(\sin bs)^{1-\mathrm{i}a/b} + z(t)(\cos bs)^{1-\mathrm{i}a/b} \tag{2.2}$$

for some constant vector c_1 and vector function $z(t)$. By substituting (2.2) into the last equation in (2.1) we have $z''(t) - \mathrm{i}\, f(t)z'(t) + z(t) = 0$. By applying these and the metric tensor of the Lagrangian surface we may obtain $|z| = |c_1|^2 = |z'(t)| = 1$ and $\langle c_1, z\rangle = \langle c_1, \mathrm{i}z\rangle = \langle z', \mathrm{i}z\rangle = 0$. Hence we obtain Case (30) after choosing suitable initial conditions.

For Case (1.ii.b.4) the following two metric tensors shall also be considered:

$$g = \mathrm{d}s \otimes \mathrm{d}s + \sinh^2(bs + \theta(t))\, \mathrm{d}t \otimes \mathrm{d}t, \tag{2.3}$$

$$g = \mathrm{d}s \otimes \mathrm{d}s + \mathrm{e}^{2bs+2\theta(t)}\, \mathrm{d}t \otimes \mathrm{d}t. \tag{2.4}$$

If (2.3) holds, Eq. (5.34) of [1] yields $\mu^2 = p^2(t)\,\mathrm{csch}^2(bs + \theta(t)) - a^2$ with $a = \sqrt{1 + b^2}$ for some $p(t) > 0$. Since $\mu = \mu(s)$, we see that $p(t)\,\mathrm{csch}(bs + \theta(t))$ depend on s. Thus $p(t)$ and $\theta(t)$ must be both constant as in Case (1.ii.c.1) of [1]. So we may assume $\theta = 0$ by applying a suitable translation. Let us denote the constant p by c. Thus the PDE system corresponding to (5.72) of [1] becomes

$$\begin{aligned} L_{ss} &= \mathrm{i}\frac{c^2\,\mathrm{csch}^2\, bs - 2a^2}{\sqrt{c^2\,\mathrm{csch}^2\, bs - a^2}} L_s - L, \\ L_{st} &= \left(\mathrm{i}\sqrt{c^2\,\mathrm{csch}^2\, bs - a^2} + b\coth bs\right) L_t, \\ L_{tt} &= \left(\mathrm{i}\sqrt{c^2 - a^2\sinh^2 bs} - b\cosh bs\right)\sinh bs L_s + \mathrm{i}\, f(t)L_t - \sinh^2 bs L \end{aligned} \tag{2.5}$$

with $a = \sqrt{b^2 + 1}$. After solving the first two equations of this system we obtain

$$\begin{aligned} L = {} & \left(a\cosh bs + \mathrm{i}\sqrt{c^2 - a^2\sinh^2 bs}\right)^{a/b} \\ & \times \left\{ \frac{z(t)\sinh bs}{\mathrm{e}^{\mathrm{i}b^{-1}c\tanh^{-1}(c\cosh bs/\sqrt{c^2 - a^2\sinh^2 bs})}} \right. \\ & \left. + c_1 \frac{\sqrt{b^2 + c^2 - \sinh^2 bs}\,\mathrm{e}^{-(\mathrm{i}a^2(b^2+c^2)/b^2(a^2+c^2))\cot^{-1}(b\cosh bs/\sqrt{c^2 - a^2\sinh^2 bs})}}{\mathrm{e}^{(\mathrm{i}c^2/b^2(a^2+c^2))\tan^{-1}(b\cosh bs/\sqrt{c^2 - a^2\sinh^2 bs})}} \right\} \end{aligned} \tag{2.6}$$

for some vector $c_1 \in \mathbf{C}^3$ and vector function $z(t)$. By substituting (2.6) into the last equation in (2.5) we obtain

$$z''(t) - \mathrm{i}\, f(t)z'(t) + (b^2 + c^2)z(t) = 0.$$

Since $|L| = |L_s| = 1$, $|L_t| = \sinh bs$ and $\langle L_s, \mathrm{i}L_y\rangle = 0$, we obtain from (2.6) that

$$|z|^2 = |c_1|^2 = \frac{b^2 + c^2}{(a^2 + c^2)^{a/b}}, \qquad |z'|^2 = \frac{1}{(a^2 + c^2)^{a/b}}, \qquad \langle z, c_1\rangle = \langle z, \mathrm{i}c_1\rangle = 0.$$

Hence the Lagrangian immersion L is congruent to Case (31).

If the metric tensor is given by (2.4), then the corresponding PDE system is given by

$$L_{ss} = \mathrm{i}\frac{c^2\,\mathrm{e}^{-2bs} - 2a^2}{\sqrt{c^2\,\mathrm{e}^{-2bs} - a^2}}L_s - L, \qquad L_{st} = (\mathrm{i}\sqrt{c^2\,\mathrm{e}^{-2bs} - a^2} + b)L_t,$$
$$L_{tt} = (\mathrm{i}\sqrt{c^2\,\mathrm{e}^{-2bs} - a^2} - b)\,\mathrm{e}^{2bs+2\theta(t)}L_s + (\mathrm{i}f(t) + \theta'(t))L_t - \mathrm{e}^{2bs+2\theta(t)}L \tag{2.7}$$

with $a = \sqrt{1 + b^2}$. After solving the first two equations of (2.7) we have

$$L = z(t)(a + \mathrm{i}\sqrt{c^2\,\mathrm{e}^{-2bs} - a^2})^{a/b}\,\mathrm{e}^{(a+b)s - \mathrm{i}b^{-1}\sqrt{c^2\,\mathrm{e}^{-2bs} - a^2}}$$
$$+ c_1\frac{(c^2 - \mathrm{e}^{2bs})(c^2 - 2\,\mathrm{e}^{2bs}(a^2 + \mathrm{i}a\sqrt{c^2\,\mathrm{e}^{-2bs} - a^2}))^{a/2b}}{(c^2 - \,\mathrm{e}^{2bs}(a^2 + b^2 + 2\mathrm{i}b\sqrt{c^2\,\mathrm{e}^{-2bs} - a^2}))^{1/2}} \tag{2.8}$$

for some vector c_1 and vector function z. By substituting (2.8) into the last equation of (2.7) we obtain

$$z''(t) - (\mathrm{i}f(t) + \theta'(t))z'(t) + c^2\,\mathrm{e}^{2\theta(t)}z(t) = 0.$$

Moreover, it follows from (2.8) and the metric tensor that

$$|z| = |c_1| = c^{-(a+b)/b}, \qquad |z'| = \mathrm{e}^{\theta(t)}\,c^{-a/b}, \qquad \langle z, c_1\rangle = \langle z, \mathrm{i}c_1\rangle = 0.$$

Therefore we may conclude that L is congruent to Case (32).

It is straightforward to verify that the three immersions defined in Cases (30)–(32) are Lagrangian surfaces of constant curvature in $CP^2(4)$.

Reference

[1] B.-Y. Chen, Classification of Lagrangian surfaces of constant curvature in complex projective planes, J. Geom. Phys. 53 (2005) 428–460.